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Spectral Density Estimation from Random Sampling for Multiplicative Stationary Processes

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Abstract—In this paper, the spectral density estimation of a nonstationary class of stochastic processes is investigated. Although these processes are not stationary with respect to the additive binary operation, i.e., in the classical weak sense, they are stationary with respect to the multiplicative binary operation. These processes exist naturally as continuous-time processes. In order to answer many questions in practical situations using these processes, we develop a random sampling method for estimating their spectral densities by using a discrete-time process. Some simulation results are given. © 2003 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, the spectral density estimation of a class of nonstationary stochastic processes is investigated. The considered nonstationary processes are referred to as multiplicative stationary processes. Although these processes are not stationary with respect to the usual additive operation, they are stationary with respect to the multiplicative operation. This property ensures essentially the same structure as is available for stationary processes, especially the spectral representation.

The concept of stationarity under a general group composition operation has been considered extensively by Hannan [1]. It may even be generalized to a semigroup operation under suitable conditions; see [2]. The multiplicative-stationary processes are defined and exist naturally as continuous-time processes. They are studied thoroughly in [3], where, especially, a duality with a stationary process is established. However, no particular advantage seems to have been taken yet of the generality afforded by this generalized concept of stationarity. A particular aspect of their study is considered here: for practical applications, a sampling of time becomes suitable for estimating the spectral density. We develop a random sampling technique, showing that the orthogonal series method for stationary processes can be used here, by way of the duality.

This method was initiated by Masry [4] to which we refer the reader for more on its qualities and advantages. It was improved by Messaci [5] under a truncated form which introduces a bandwidth parameter allowing a better adjustment of the estimate. Consistency in mean-square of the estimate for the multiplicative-stationary case is established here. Numerical examples are also given.

This paper is organized as follows. In Section 2, we recall some properties and examples of M-stationary processes. The sampling technique and the linked spectral density estimate construction are exposed in Section 3. We give the results of convergence of the estimate in Section 4. A simulation study is given in Section 5. And a wider concept of stationarity is introduced in Section 6, extending the multiplicative stationarity.

2. SOME PROPERTIES OF MULTIPLICATIVE-STATIONARY PROCESSES

Let \mathbb{E} denote the expectation operator and $t \cdot \tau$ the usual multiplication in \mathbb{R} .

DEFINITION 1. Let $X = \{X(t), t > 0\}$ be a real-valued stochastic process such that, for all positive real numbers t and τ ,

- (i) $\mathbb{E}\{X(t)\} = \mu$ is independent of t ,
- (ii) $\text{Var}\{X(t)\} < +\infty$,
- (iii) $\mathbb{E}\{(X(t) - \mu)(X(t \cdot \tau) - \mu)\} = R_X(\tau)$ exists and is independent of t .

Then X is said to be a weakly multiplicative-stationary (or M-stationary) process.

In the classical sense, M-stationary processes are nonstationary, but each M-stationary process corresponds to a stationary process as stated in the following theorem due to Gray and Zhang [3].

THEOREM 1. Let $Y = \{Y(u), u \in \mathbb{R}\}$ be a stochastic process such that $Y(u) = X(\exp u)$. Then X is weakly M-stationary if and only if Y is weakly stationary.

Y is referred to as the dual stationary process of X . Let us denote by R_Y the autocovariance function of Y ; then for all positive real number τ , we have

$$R_X(\tau) = R_X(\tau^{-1}) = R_Y(\ln \tau).$$

If we assume that R_X is continuous and that $\tau^{-1/2}R_X(\tau) \in L^2(\mathbb{R}_+)$ (respectively, $\tau^{-1}R_X(\tau) \in L^1(\mathbb{R}_+)$), then R_Y is continuous and square integrable (respectively, absolutely integrable).

As seen in the following, this duality is the basis of the development of the theory of spectral density estimation of an M-stationary process.

First, specializing [6] results to the case of M-stationary processes, we get the following condition for a process to be M-stationary.

THEOREM 2. Let $X = \{X(t), t \in \mathbb{R}_+^*\}$ be a zero-mean stochastic process such that the autocovariance function $r_X(t, t') = \mathbb{E}\{X(t)X(t')\}$ is finite for all positive real numbers t and t' . Then there exists a function R such that $r_X(t, t \cdot \tau) = R(\tau)$ if and only if

$$\frac{\partial r_X(t, t \cdot \tau)}{\partial x} = -\tau \cdot \frac{\partial r_X(t, t \cdot \tau)}{\partial y}, \quad (1)$$

for all positive real numbers t and τ .

PROOF. The direct implication is obvious.

Conversely, set

$$\begin{array}{ll} u = \ln \tau, & \text{or} \quad t = e^{(u-u')/2}, \\ u' = \ln t^2 \tau, & \tau = e^u, \end{array}$$

and $\Gamma(u, u') = r_X(e^{(u-u')/2}, e^u)$. By (1), we get

$$\frac{\partial \Gamma(u, u')}{\partial y} = 0,$$

and hence, $\Gamma(u, u') = R(u) = R(\ln \tau) = r_X(t, t \cdot \tau)$. And the result follows. ■

Let us give some examples of M-stationary processes to set up ideas.

EXAMPLE 1. Consider the process X , defined for $t > 0$ by

$$X(t) = \sum_{i=1}^N [A_i \cos(a_i \ln t) + B_i \sin(a_i \ln t)],$$

where A_i and B_i are zero-mean uncorrelated random variables such that $\mathbb{E}\{A_i^2\} = \mathbb{E}\{B_i^2\}$ for all $i \in \{1, \dots, n\}$. Then X is a zero-mean M-stationary process with

$$R_X(\tau) = \sum_{i=1}^N \mathbb{E}\{A_i^2\} \cos(a_i \ln \tau). \quad \blacksquare$$

EXAMPLE 2. A process ε is said to be an M-white noise if its dual process is a white noise. An M-white noise is thus defined by

$$\mathbb{E}\{\varepsilon(t)\} = 0 \quad \text{and} \quad \mathbb{E}\{\varepsilon(t)\varepsilon(t \cdot \tau)\} = c \delta(\ln \tau),$$

where δ is the Kronecker symbol. ■

EXAMPLE 3. Consider the process X , defined for $t > 0$ by $X(t) = \int_0^t G(1/u) \varepsilon(u) d \ln u$, where

- $\int_1^{+\infty} |G(t)|^2 d \ln t > -\infty$ and $G(t) = 0$ for $0 < t < 1$;
- ε is an M-white noise.

Then X is an M-stationary process called an M-linear process. ■

EXAMPLE 4. The Euler process is a particular case of M-linear processes, sometimes called the long memory process with

$$G(t) = \sum_{i=1}^M \sum_{j=1}^{m_i} c_{ij} (\ln t)^j t^{-a_i}, \quad \text{for } t \geq 1,$$

where c_{ij} and a_i are complex numbers such that $\operatorname{Re}(a_i) \geq 0$, where $\operatorname{Re}(z)$ denotes the real part of the complex number z . We have

$$R_X(\tau) = \sum_{i=1}^M \sum_{j=1}^{m_i} b_{ij} (\ln \tau)^j \tau^{-a_i},$$

where the constants b_{ij} are uniquely determined from the constants a_i and c_{ij} .

The integer $k = \sum_{i=1}^M (m_i + 1)$ is called the order of the process. Note that the Euler processes play the same part in the M-stationary processes theory as the autoregressive stationary models do in the classical stationary process theory. ■

Let X be an M-stationary process. The M-spectral density H_X of X is defined as the Mellin transform of the autocovariance R_X if it does exist, namely

$$H_X(\lambda) = \int_0^{+\infty} \tau^{-2i\pi\lambda} R_X(\tau) d \ln \tau. \quad (2)$$

It may also be written as

$$H_X(\lambda) = \int_{-\infty}^{+\infty} e^{-2i\pi\lambda u} R_Y(u) du,$$

where R_Y is the autocovariance function of the dual process Y of X .

We have implicitly assumed in (2) that the Mellin transform exists. We also make this assumption for the inverse Mellin transform. For details on the Mellin transform, see [7] and the references therein. From (2), it follows at once that $H_X(\lambda) = H_Y(\lambda)$, where H_Y is the spectral density of Y . Hence, we shall use the term spectral density for M-spectral density.

In the sequel, we shall assume that the process X is a zero-mean real valued process in order to simplify the notation. For developing the estimation method, we need also to assume that X is M-stationary up to order four, i.e., that $\mathbb{E}\{X(t)X(t \cdot t_1)X(t \cdot t_2)X(t \cdot t_3)\}$ is independent of t , for all positive real numbers t_1, t_2, t_3 , and t . In this case, we call M-cumulant of order four of X , the function

$$\begin{aligned} C_X(t_1, t_2, t_3) &= \text{Cum}\{X(t), X(t_1), X(t_2), X(t_3)\} \\ &= \mathbb{E}\{X(t)X(t \cdot t_1)X(t \cdot t_2)X(t \cdot t_3)\} - R_X(t_1)R_X\left(\frac{t_2}{t_3}\right) \\ &\quad - R_X(t_2)R_X\left(\frac{t_3}{t_1}\right) - R_X(t_3)R_X\left(\frac{t_2}{t_1}\right). \end{aligned}$$

If $Y = \{Y(u), u \in \mathbb{R}\}$ is the dual process of X , it is zero mean, stationary in the classical sense, of order four, and with cumulant function C_Y given by

$$\begin{aligned} C_Y(u_1, u_2, u_3) &= C_X(\exp(u_1), \exp(u_2), \exp(u_3)) \\ &= \text{Cum}\{Y(u), Y(u + u_1), Y(u + u_2), Y(u + u_3)\}. \end{aligned}$$

3. TIME SAMPLING AND SPECTRAL DENSITY ESTIMATION

For sampling time, we consider the scheme $\{t_n\}_{n \in \mathbb{N}}$ proposed by Messaci [5], namely

$$t_0 = 1 \quad \text{and} \quad t_n = t_{n-1}\beta_n, \quad \text{for } n \geq 1, \quad (3)$$

where $\{\beta_n\}_{n \in \mathbb{N}}$ is a sequence of independent identically Pareto distributed random variables. It has the probability density

$$f_{\beta_n}(x) = \beta x^{-\beta-1} \mathbf{1}_{[1, +\infty[}(x), \quad \beta > 0.$$

Note that since $\beta_n \geq 1$ a.s., we also have $t_n \geq 1$ a.s.

Consider then the scheme $\tau_n = \ln t_n$ for all $n \in \mathbb{N}$. The sequence $\{\tau_n\}_{n \in \mathbb{N}}$ satisfies $\tau_0 = 0$ and $\tau_n = \tau_{n-1} + \alpha_n$ for $n \in \mathbb{N}^*$, where $\alpha_n = \ln \beta_n$ is a sequence of independent identically exponentially distributed random variables with parameter $\beta > 0$.

For any $k \geq 1$, the difference $\tau_{n+k} - \tau_k = \sum_{i=1}^n \alpha_{k+i}$ is a Gamma distributed random variable, with order n and parameter β . It has the probability density

$$f_n(t) = \beta \frac{(\beta t)^{n-1}}{(n-1)!} \exp(-\beta t) \mathbf{1}_{\mathbb{R}_+}(t).$$

Let $X = \{X(t), t \geq 0\}$ be an M-stationary process and let $Y = \{Y(u), u \in \mathbb{R}\}$ be its dual process. Consider the discrete time process $\{X(t_n)\}_{n \in \mathbb{N}}$. This process is of zero mean. We have $X(t_n) = Y(\tau_n)$ and $R_X(t_n) = R_Y(\ln t_n) = R_Y(\tau_n)$. Since the process X and the sequence $\{t_n\}$ are independent, the autocovariance function of $\{X(t_n)\}$ can be written as

$$\begin{aligned} \rho_X(n) &= \mathbb{E}\{X(t_{k+n})X(t_k)\} = \mathbb{E}_{\{t_n\}}[\mathbb{E}_X\{X(t_{k+n})X(t_k)\}] \\ &= \mathbb{E}_{\{t_n\}}\left\{R_X\left(\frac{t_{k+n}}{t_k}\right)\right\} = \mathbb{E}_{\{t_n\}}\{R_Y(\tau_{n+k} - \tau_k)\} \\ &= \int_{\mathbb{R}_+} R_Y(t) f_n(t) dt, \end{aligned}$$

where $\mathbb{E}_{\{t_n\}}$ (respectively, \mathbb{E}_X) denotes the conditional expectation with respect to the distribution of $\{t_n\}$ (respectively, of X).

Moreover, the system $\{f_n\}_{n \in \mathbb{N}^*}$ is complete in $L^2(\mathbb{R}_+)$ and is orthonormalized as

$$g_n(t) = \sqrt{2\beta} L_{n-1}^{(0)}(2\beta t) \exp(-\beta t) \mathbf{1}_{\mathbb{R}_+}(t), \quad (4)$$

where $L_n^{(0)}$ is the n^{th} Laguerre polynomial. For details on Laguerre polynomials, see [8].

Conversely, the expansion of $g_n(t)$ in the basis $\{f_k(t)\}_{k=1}^n$ is

$$g_n(t) = \sum_{k=1}^n \theta_{n,k} f_k(t), \quad \text{where } \theta_{n,k} = \sqrt{\frac{2}{\beta}} (-2)^{k-1} C_{n-1}^{k-1}.$$

Since the process X is real valued, we obtain $R_X(t) = \sum_{n=1}^{+\infty} a_n g_n(|t|)$ in $L^2(\mathbb{R})$, with

$$a_n = \int_{\mathbb{R}_+} R_Y(t) g_n(t) dt = \sum_{k=1}^n \theta_{n,k} \int_{\mathbb{R}_+} R_Y(t) f_k(t) dt = \sum_{k=1}^n \theta_{n,k} \rho_X(k),$$

and the spectral density can be written as

$$H_X(\lambda) = \int_{\mathbb{R}} \exp(-2i\pi\lambda u) R_Y(u) du = \sum_{n=1}^{+\infty} a_n G_n(\lambda),$$

with

$$G_n(\lambda) = \int_{\mathbb{R}} \exp(-2i\pi\lambda u) g_n(|u|) du = \int_{\mathbb{R}} \exp(-2i\pi\lambda u) \sum_{k=1}^n \theta_{n,k} f_k(|u|) du.$$

Since the function f_k is the probability density of a Gamma distribution with order k and parameter β , its characteristic function is

$$\phi(\lambda) = \left(\frac{\beta}{\beta - i\lambda} \right)^k.$$

We compute

$$\begin{aligned} G_n(\lambda) &= 2\mathcal{R}e \left[\sum_{k=1}^n \theta_{n,k} \left(\frac{\beta}{\beta - 2i\pi\lambda} \right)^k \right] \\ &= \frac{2(-1)^{n-1}}{\pi} \sqrt{\frac{2\beta}{\beta^2 + 4\pi^2\lambda^2}} \cos \left[(2n-1) \arctan \left(\frac{2\pi\lambda}{\beta} \right) \right]. \end{aligned}$$

Note that $\int_{\mathbb{R}} G_n(\lambda) G_p(\lambda) d\lambda = 2\delta_{n,p}$, where $\delta_{n,p}$ is the Kronecker symbol.

The expansion of the spectral density in the basis $\{G_n\}_{n \in \mathbb{N}^*}$ is $H_X(\lambda) = \sum_{n=1}^{+\infty} a_n G_n(\lambda)$, with $a_n = \sum_{k=1}^n \theta_{n,k} \rho_X(k)$ and $\theta_{n,k} = \sqrt{2/\beta} (-2)^{k-1} C_{n-1}^{k-1}$.

For estimating H_X , let us be given the N observations $X(t_k)$, $k = 1, \dots, N$. First, $\rho_X(n)$ may be estimated by

$$\rho_{x,N}^{(n)} = \begin{cases} \frac{1}{N} \sum_{k=1}^{N-n} X(t_{k+n}) X(t_k), & \text{if } 1 \leq n < N, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we estimate a_n by

$$\hat{a}_n(N) = \sum_{k=1}^n \theta_{n,k} \rho_{x,N}^{(k)}.$$

And the spectral density estimate is finally given by

$$\hat{H}_{X,N}(\lambda) = \sum_{n=1}^{M_N} y_n(\lambda) \hat{a}_n(N) G_n(\lambda), \quad (5)$$

with the bandwidth $M_N = [b \ln N / 2\alpha] + 1$ and with $y_n(\lambda) = h(\exp(\alpha n) / N^b)$, where $\alpha > 0$ and $0 < b < \alpha / \ln 3$, and where the real function h is a Lipschitz function of order 1 such that $h(u) \leq h(0) = 1$, for all $u \in \mathbb{R}$.

4. ASYMPTOTIC PROPERTIES OF THE ESTIMATE

For studying the asymptotic behaviour of the spectral density estimate $\hat{H}_{X,N}$ given in (5), we need the following conditions.

CONDITION 1. Let $AC^{r-1}([1, +\infty[)$ denote the set of r -times differentiable functions with absolutely continuous successive derivatives. We assume that there exists an $r \geq 3$ such that $R_X \in AC^{r-1}([1, +\infty[)$ and that

$$[t^{2k-1} \ln^r t]^{1/2} R_X^{(k)}(t) \in L^2([1, +\infty[), \quad \text{for } k \in \{0, \dots, r\}.$$

CONDITION 2. $(\ln t/t)^{1/2} R_X(t) \in L^2([1, +\infty[)$ and $|C_X(t_1, t_2, t_3)| \leq h_1(t_1, t_2, t_3)$, for all positive numbers t_1, t_2 , and t_3 , where h_1 is a nonincreasing function on $[1, +\infty[$ for each argument verifying

$$h_1(t_1, t_2, t_3) = h_1\left(\frac{1}{t_1}, t_2, t_3\right) = h_1\left(t_1, \frac{1}{t_2}, t_3\right) = h_1\left(t_1, t_2, \frac{1}{t_3}\right)$$

and

$$\int_1^{+\infty} h_1(1, \tau, 1) d \ln \tau < +\infty.$$

Note that Conditions 1 and 2 are equivalent to those assumed by Masry [4] or Messaci [5] for the stationary process Y dual to X , since by duality, $R_X(t) = R_Y(\ln t)$.

In order to study the behavior of the estimate $\hat{H}_{X,n}$, we have to study first the behavior of the estimates \hat{a}_n and $\hat{\rho}_{X,n}$. This will be done in the two following propositions.

The first result concerns $\hat{\rho}_{X,n}$.

PROPOSITION 1. Under Condition 1, $\hat{\rho}_{X,n}$ is consistent in mean-square, that is to say the following.

$$(1) \quad \mathbb{E}\{\hat{\rho}_{X,N}(n)\} = (1 - n/N)\rho_X(n).$$

(2) There exists a uniform positive constant A_2 on n and N such that

$$\text{Var}\{\hat{\rho}_{X,n}(N)\} \leq \frac{A_2}{N}.$$

PROOF.

(1) We have

$$\mathbb{E}\{\hat{\rho}_{X,N}(n)\} = \frac{1}{N} \sum_{k=1}^{N-n} \mathbb{E}\{X(t_{k+n}) - X(t_k)\} = \left(1 - \frac{n}{N}\right) \rho_n.$$

Thus, the order of magnitude of the bias of $\hat{\rho}_{X,n}$ is $O(1/N)$.

(2) We have

$$\text{Var}\{\hat{\rho}_{X,N}(n)\} = \frac{1}{N^2} \sum_{k,l=1}^{N-n} \text{Cov}\{X(t_{k+n})X(t_k), X(t_{l+n})X(t_l)\}.$$

With some simple algebra, this becomes

$$\begin{aligned} \text{Var}\{\hat{\rho}_{X,N}(n)\} &= \frac{1}{N^2} \sum_{k,l=1}^{N-n} \mathbb{E}_{\{t_n\}} \{R_Y(t_{k+n} - t_k)R_Y(t_{l+n} - t_l) + R_Y(t_l - t_k)R_Y(t_{l+n} - t_{k+n}) \\ &\quad + R_Y(t_{k+n} - t_k)R_Y(t_{k+n} - t_l) + C_X(t_{k+n}, t_k, t_{l+n}, t_l)\}. \end{aligned}$$

The stationarity up to order four of the process induces that

$$\begin{aligned} \text{Cum}\{X(t_{k+n}), X(t_k), X(t_{l+n}), X(t_l)\} &= C_X\left(\frac{t_{k+n}}{t_k}, \frac{t_{l+n}}{t_k}, \frac{t_l}{t_k}\right) \\ &= C_Y(t_{k+n} - t_k, t_{l+n} - t_k, t_l - t_k). \end{aligned}$$

Hence, the technique used in [9, pp. 17–25], applied here to the stationary process dual to X ensures that $\text{Var}\{\hat{\rho}_{X,N}(n)\} = O(1/N)$. ■

The estimate \hat{a}_n of a_n behaves as follows.

PROPOSITION 2. Under Condition 2, we have the following.

- (1) $\mathbb{E}\{\hat{a}_n(N)\} = a_n - (na_n - (n-1)a_{n-1})/N$.
- (2) $\text{Var}\{\hat{a}_n(N)\} \leq A_3 3^{2n}/N$, where $A_3 = 2A_2/9\beta$.

PROOF. Recall that $\hat{a}_n(N) = \sum_{k=1}^n \theta_{n,k} \hat{\rho}_{X,k}(N)$. Hence, we deduce both points (1) and (2) from Theorem 3.

(1) We have

$$\mathbb{E}\{\hat{a}_n(N)\} = \sum_{k=1}^n \theta_{n,k} \mathbb{E}\{\hat{\rho}_{X,N}(k)\} = a_n - \frac{1}{N} \sum_{k=1}^n \theta_{n,k} \rho_X(k) \frac{k}{N} = a_n - \text{Bias}(\hat{a}_n(N)),$$

where $\text{Bias}(\hat{a}_n(N))$ denotes the bias of $\hat{a}_n(N)$. We then get

$$\text{Bias}\{\hat{a}_n(N)\} = -\frac{1}{N} \sum_{k=1}^n \theta_{n,k} k \int_0^{+\infty} R_Y(t) f_k(t) dt = \frac{-1}{N} \int_0^{+\infty} \left(\sum_{k=1}^n \theta_{n,k} k f_k(t) \right) R_Y(t) dt.$$

But $\sum_{k=1}^n k \theta_{n,k} f_k(t) = ng_n(t) - (n-1)g_{n-1}(t)$, with $g_n(t)$ defined in (4). Thus,

$$\begin{aligned} \text{Bias}\{\hat{a}_n(N)\} &= \frac{-n}{N} \int_0^{+\infty} R_Y(t) g_n(t) dt + \frac{n-1}{N} \int_0^{+\infty} R_Y(t) g_{n-1}(t) dt \\ &= \frac{-n}{N} a_n + \frac{n-1}{N} a_{n-1}. \end{aligned}$$

(2) The autocovariance function of the estimate $\hat{a}_n(N)$ can be bounded as follows:

$$\text{Var}\{\hat{a}_n(N)\} \leq \left(\sum_{k=1}^n |\theta_{n,k}| \sqrt{\text{Var}\{\hat{\rho}_{X,n}(N)\}} \right)^2.$$

From Theorem 3, this becomes, by some simple manipulations,

$$\text{Var}\{\hat{a}_n(N)\} \leq \left(\sum_{k=1}^n \sqrt{\frac{2A_2}{\beta N}} 2^{k-1} C_{n-1}^{k-1} \right)^2 \leq 3^{2n} \frac{A_3}{N},$$

with $A_3 = 2A_2/9$. This completes the proof. ■

The mean-squared error of the estimate $\hat{H}_{X,n}$ is given in the following theorem.

THEOREM 3. If both Conditions 1 and 2 are satisfied, we have

$$\mathbb{E}\left\{\hat{H}_{X,N}(\lambda) - H_X(\lambda)\right\}^2 = O\left(\frac{1}{\ln N}\right)^{r-2}.$$

PROOF. The mean-squared error of $\hat{H}_{X,n}$ may be written as

$$\mathbb{E}\left\{\hat{H}_{X,N}(\lambda) - H_X(\lambda)\right\}^2 = \text{Var}\left\{\hat{H}_{X,N}(\lambda)\right\} + \text{Bias}\left(\hat{H}_{X,N}(\lambda)\right)^2.$$

The proof of the theorem will be achieved by setting upper bounds for both terms of the sum.

FIRST STEP. Bound for the variance.

From the expression of the estimate and Theorem 4, we get

$$\begin{aligned} \left(\text{Var}\left\{\hat{H}_{X,N}(\lambda)\right\}\right)^{1/2} &\leq \sum_{n=1}^{M_N} |y_n(\lambda)| |G_n(\lambda)| \sqrt{\frac{A_3 3^{2n}}{N}} \\ &\leq 2 \sum_{n=1}^{M_N} h\left(\frac{\exp(na)}{N^b}\right) \sqrt{\frac{2A_3 \beta 3^{2n}}{N(\beta^2 + 4\pi^2 \lambda^2)}}. \end{aligned} \tag{6}$$

From the expression of the bandwidth M_N , the quantity in (6) is less than

$$9N^{-1/2(1-b \ln 3/2\alpha)} \sqrt{\frac{2A_3}{\beta}},$$

and thus, $\text{Var} \{\hat{H}_{X,N}(\lambda)\} = O(N^{-p})$ with $p = 1 - b \ln 3/2\alpha$.

SECOND STEP. Bound for the bias.

We have

$$\begin{aligned} \text{Bias} \left\{ \hat{H}_{X,N}(\lambda) \right\} &= \sum_{n=1}^{M_N} y_n(\lambda) G_n(\lambda) a_n - \sum_{n=1}^{M_N} a_n G_n(\lambda) - \sum_{n=M_N+1}^{+\infty} a_n G_n(\lambda) \\ &\quad - \frac{1}{N} \sum_{n=1}^{M_N} G_n(\lambda) y_n(\lambda) [n a_n - (n-1) a_{n-1}], \end{aligned}$$

which implies

$$\begin{aligned} \left| \text{Bias} \left\{ \hat{H}_{X,N}(\lambda) \right\} \right| &\leq \sum_{n=1}^{M_N} |G_n(\lambda) a_n (1 - y_n(\lambda))| + \frac{1}{N} \sum_{n=1}^{M_N} |y_n(\lambda) G_n(\lambda)| |n a_n + (n-1) a_{n-1}| \\ &\quad + \sum_{n=1}^{M_N} |a_n G_n(\lambda)|. \end{aligned}$$

We will now bound the three terms of this sum separately.

(i) We have

$$\sum_{n=1}^{M_N} |G_n(\lambda) a_n (1 - y_n(\lambda))| \leq \sqrt{\frac{2}{\pi^2 \beta}} \sum_{n=1}^{M_N} |a_n| |1 - y_n(\lambda)|.$$

Lemma 2.1 of [10] states that under both Conditions 1 and 2,

$$|a_n| \leq A_1(r) n^{-r/2}, \quad (7)$$

where

$$A_1(r) = \sqrt{\frac{1}{2\beta}} \left\| t^{r/2} \frac{d^r}{dt^r} \left[R_Y \left(\frac{t}{2\beta} \right) \exp \left(-\frac{t}{2} \right) \right] \right\|_{L^2(\mathbb{R}_+)}.$$

The proof is based on the properties of Laguerre polynomials. Using both inequality (7) and the Lipschitz properties of h , we obtain

$$\sum_{n=1}^{M_N} |G_n(\lambda) a_n [1 - y_n(\lambda)]| \leq k_2 A_1(r) \sqrt{\frac{2}{\pi^2 \beta}} \sum_{n=1}^{M_N} n^{-r/2} \frac{\exp(n\alpha)}{N^b}.$$

From

$$\sum_{n=1}^{M_N} n^{-r/2} \leq 1 + \int_1^{+\infty} x^{-r/2} dx \leq \frac{r}{r-2}$$

and $M_N \leq b(\ln N + 1)/2\alpha$, we deduce finally that

$$\sum_{n=1}^{M_N} |G_n(\lambda) a_n [1 - y_n(\lambda)]| \leq \sqrt{\frac{2}{\pi^2 \beta}} k_2 A_1(r) \frac{r}{r-2} e N^{-b/2}.$$

The right-hand term of this inequality is of order of magnitude $O(1/N^{b/2}) = O[(1/\ln N)^{(r-2)/2}]$.

(ii) By some simple algebra, we get

$$\begin{aligned} & \frac{1}{N} \sum_{n=1}^{M_N} |y_n(\lambda) G_n(\lambda)| |na_n + (n-1)a_{n-1}| \\ & \leq \frac{2}{N} \sqrt{\frac{2}{\pi^2 \beta}} \sum_{n=1}^{+\infty} M_N A_1(r) n^{-r/2} \leq 2A_1(r) \sqrt{\frac{2}{\pi^2 \beta}} \sum_{n=1}^{+\infty} n^{-r/2} \left(\frac{b}{2a} \ln N + 1 \right) \frac{1}{N^{-p+1} N^p}, \end{aligned}$$

where $p = 1 - b \ln 3 / \alpha$. The conditions imposed on a and b ensure that $0 < p < 1$, which implies first that $(1 + b \ln N / 2\alpha) / N^{-p+1} = o(1)$, and second that

$$\frac{1}{N} \sum_{n=1}^{M_N} |y_n(\lambda) G_n(\lambda)| |na_n + (n-1)a_{n-1}| = O\left(\frac{1}{N}\right)^p = O\left[\left(\frac{1}{\ln N}\right)^{(r-2)/2}\right].$$

(iii) Using inequality (7) again, we get

$$\begin{aligned} \sum_{n=M_N+1}^{+\infty} |a_n G_n(\lambda)| & \leq \sqrt{\frac{2}{\pi^2 \beta}} A_1(r) \sum_{n=M_N+1}^{+\infty} n^{-r/2} \leq \sqrt{\frac{2}{\pi^2 \beta}} A_1(r) \int_{M_N}^{+\infty} x^{-r/2} dx \\ & \leq \sqrt{\frac{2}{\pi^2 \beta}} A_1(r) \frac{2}{r-2} M_N^{(2-r)/2} \leq \sqrt{\frac{2}{\pi^2 \beta}} A_1(r) \left(\frac{2\alpha}{b \ln N}\right)^{(r-2)/2}. \end{aligned}$$

Thus,

$$\sum_{n=M_N+1}^{+\infty} |a_n G_n(\lambda)| = O\left[\left(\frac{1}{\ln N}\right)^{(r-2)/2}\right].$$

From the three substeps above, we deduce directly the result. ■

Finally, the mean-squared error of $\hat{H}_{X,N}$ is given in the following theorem.

THEOREM 4. *Under the same conditions, the integrated mean-squared error of the estimate $\hat{H}_{X,n}$ satisfies*

$$\mathbb{E} \left\{ \int_{-\infty}^{+\infty} |\hat{H}_{X,n}(\lambda) - \phi(\lambda)|^2 d\lambda \right\} = O\left(\frac{1}{\ln N}\right)^{r-1}.$$

PROOF. The proof is induced by a simple integration from Theorem 5. ■

The asymptotic normality of the estimate may be established. Suppose X is M-stationary up to all order k and if the k^{th} order cumulant of X ,

$$C_X^{(k)}(t_1, \dots, t_{k-1}) = \text{Cum} \{X(t \cdot t_1), \dots, X(t \cdot t_{k-1}), X(t)\},$$

is integrable. Then, under the assumptions of Theorem 5, the variables $(\ln N)^{r/2-1}[\hat{H}_{X,n}(\lambda_i) - \mathbb{E}\{\hat{H}_{X,n}(\lambda_i)\}]$ are joint normally distributed. This is equivalent to showing that all their joint cumulants of order $k \geq 3$ tend to zero as N tends to infinity. The proof can be adapted from the proof of the same result for stationary processes made in [11].

5. SIMULATION STUDY

The calculus of the defined estimates is based on some simulated samples. Let X be a second-order M-stationary Gaussian process with zero mean, continuous autocovariance function $R_X \in L^1$, and spectral density function H_X . We study the finite-sample size performance of the spectral estimate (5), through the Monte-Carlo simulation, for both following autocovariance and spectral density functions

$$R_X(t) = \exp(-\ln t) \quad \text{and} \quad H_X(\lambda) = \frac{1}{\pi(1 + \lambda^2)}, \quad (8)$$

and

$$R_X(t) = \exp(-a \ln t) \left(\cos \omega_0 \ln t - \frac{a}{\omega_0} \sin \omega_0 \ln t \right) \quad \text{and} \quad H_X(\lambda) = \frac{2a\lambda^2}{\pi [(\lambda^2 - \lambda_0^2) + 4a^2\lambda_0^2]}, \quad (9)$$

where $\omega_0 = \lambda_0^2 - a^2$.

Note that H_X is the lowpass spectral density of the Gauss-Markov process in equation (8) and a narrowband spectral density centered at λ_0 in equation (9). These examples are taken from [4].

We will use two samples of respective sizes $N = 500$ and $N = 1000$, and consider Parzen's spectral window (see [12]), for which

$$h(t) = \begin{cases} 1 - 6t^2 + 6|t|^3, & \text{if } |t| \leq \frac{1}{2}, \\ 2(1 - |t|)^3, & \text{if } \frac{1}{2} < |t| \leq 1, \\ 0, & \text{if } 1 < |t|. \end{cases}$$

The estimate was computed for several values of the sampling rate β in order to test its inference on the performance of the estimate. For each sampling rate, the estimate (5) was initially computed for several values of M_N . For each estimate, the optimal M_N was chosen following the cross-validation procedure developed in [13]: the bandwidth M_N minimizing a cross-validated criterion depending only on the observations is proved to minimize asymptotically the integrated mean-squared error between H_X and $\hat{H}_{X,N}$ over the range of graphs. The simulation results indicate that the sample size performance of estimate (5) is very good, that is to say, insensitive to the sampling rate β . Hence, we give the results here only for one sampling rate, namely $\beta = 1/\pi$.

The sampling times $\{t_k\}_{k=1}^N$ and the observations $\{X(t_k)\}_{k=1}^N$ are generated recursively as follows. The $\{t_k\}_{k=1}^N$ are already given by (3) in a recursive form. Sample values of the $\{\alpha_k\}_k$ are obtained as

$$\alpha_k = -\frac{1}{\beta} \ln \theta_k, \quad k = 1, \dots, N,$$

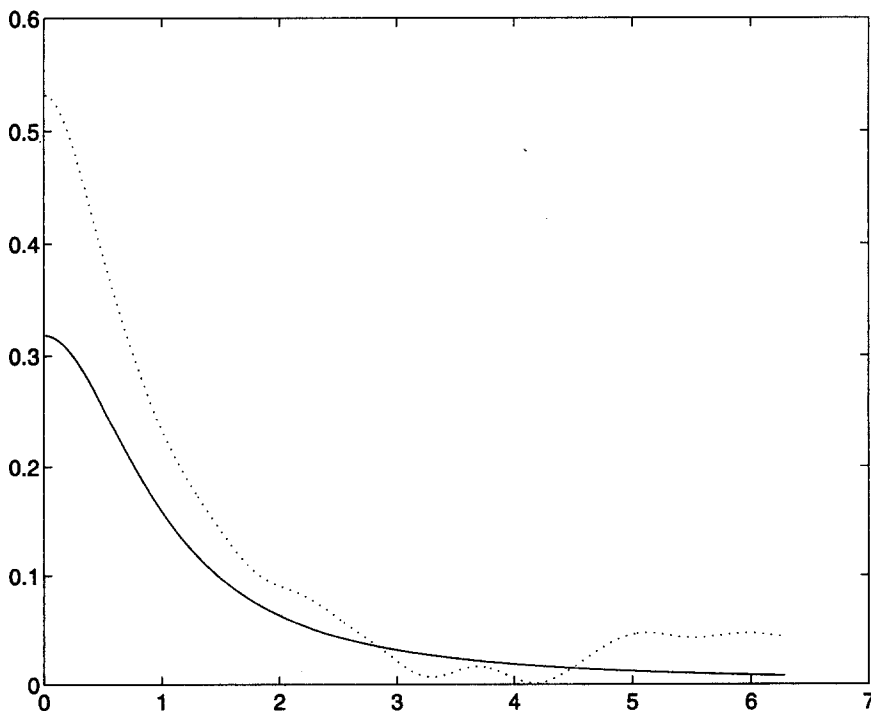


Figure 1. Gauss-Markov spectrum and its estimate: $\beta = 1/\pi$, $N = 500$.

where the $\{\theta_k\}_k$ are uniform on $[0, 1]$ random numbers. The observations $\{X(t_k)\}_{k=1}^N$ are then obtained via the following dynamical representation for the process X (see [14]):

$$X(t) = \exp[-(t - \tau)]X(\tau) + \sqrt{2} \int_{\tau}^t \exp[-\tau(t - s)] dW(s), \quad t > \tau \geq 0,$$

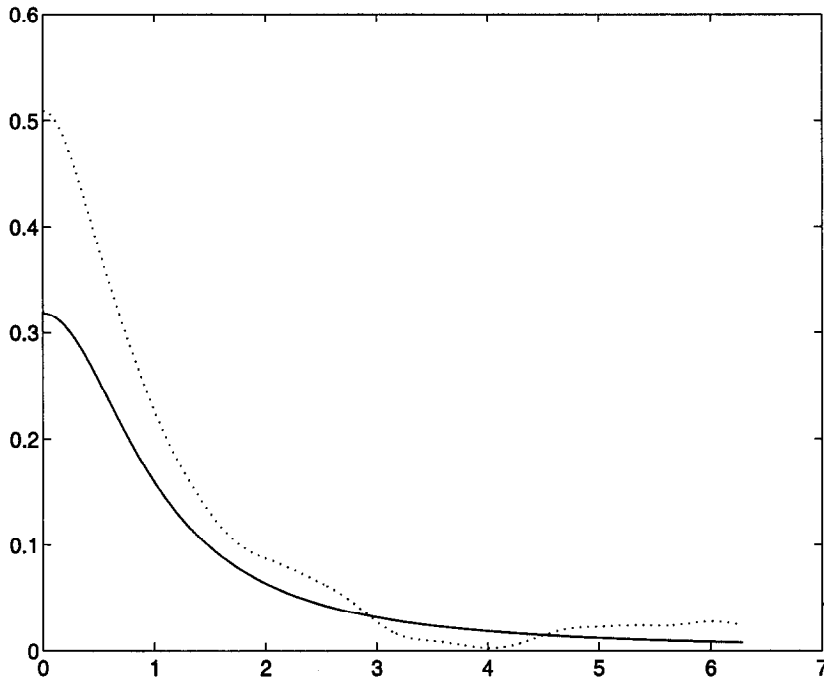


Figure 2. Gauss-Markov spectrum and its estimate: $\beta = 1/\pi$, $N = 1000$.

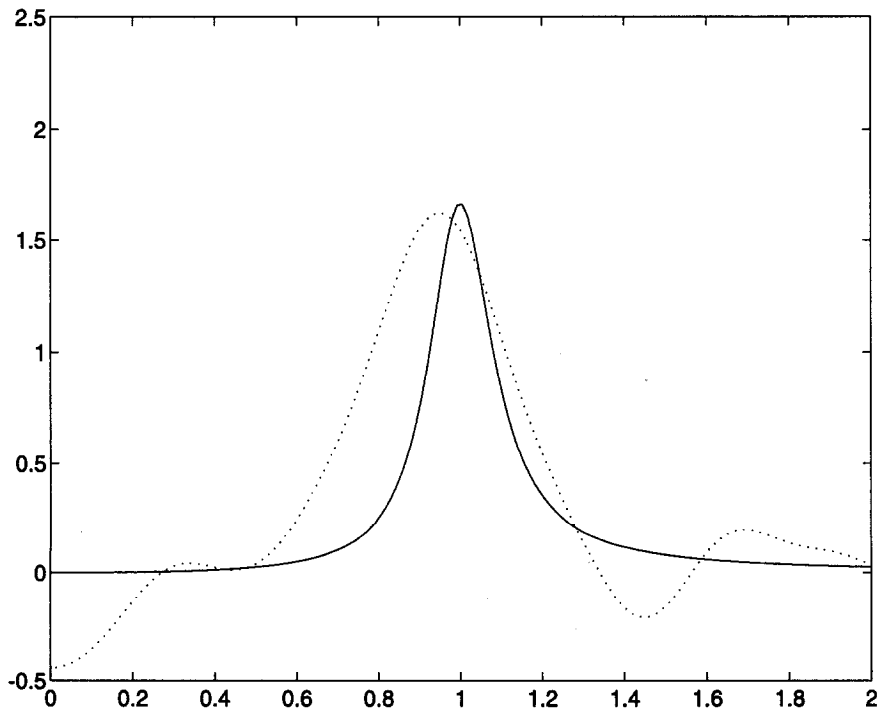


Figure 3. Narrowband spectrum and its estimate: $\lambda_0 = 1$, $\alpha = \beta = 1, 2/22$, $N = 500$.

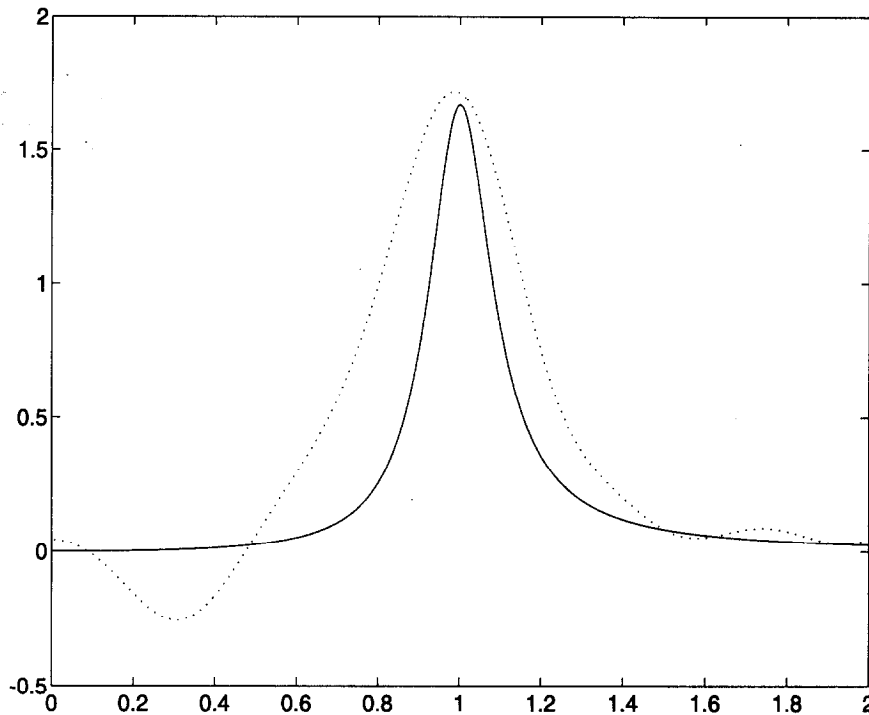


Figure 4. Narrowband spectrum and its estimate: $\lambda_0 = 1$, $a = \beta = 1, 2/22$, $N = 1000$.

where $\{W(t), t \geq 0\}$ is the real valued Wiener process with zero mean and autocovariance function $R_X(t, s) = \min(t, s)$.

Then, we easily get $X(t_0) = 0$, and

$$X(t_{k+1}) = \exp[-(t_{k+1} - t_k)]X(t_k) + (1 - \exp[-2(t_{k+1} - t_k)])^{1/2} \gamma_{k+1}, \quad k \geq 1,$$

where the γ_k are i.i.d. standard Gaussian random variables.

Sample values of $\{\gamma_k\}_k$ are obtained from a Gaussian random number generator. The estimate (5) is then computed. The results shown in Figures 1–4 confirm that the estimate performs well.

6. CONCLUSION

The bandwidth selection is an important problem. In fact, an unsuitable selection of this parameter (so not minimizing optimally the mean-squared error) induces a slow rate of convergence of the estimates. Techniques such as the cross-validation developed in [13] for continuous time stationary process (and used here in the simulation study) can solve this problem.

Spectral density estimation for nonstationary processes is not uniquely linked to the definition of the multiplicative stationarity. It is extended in [15] to other generalized forms of stationarity, where a suitable time transformation is used instead of the logarithm. The spectral density is then deduced from the corresponding theory of group representation coming in place of the Mellin transform. Examples of such processes are given, especially linear processes, and their extensive study is developed.

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